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Oxidizing super–Yang-Mills from ($N = 4$, $d = 4$) to ($N = 1$, $d = 10$)

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Abstract: We introduce superspace generalizations of the transverse derivatives to rewrite the four-dimensional $N = 4$ Yang-Mills theory into the fully ten-dimensional $N = 1$ Yang-Mills in light-cone form. The explicit SuperPoincaré algebra is constructed and invariance of the ten-dimensional action is proved.

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1. Introduction

Superstring theories are tantalizingly close to the real world, in the sense that they include both gravitational and Yang-Mills interactions, but they suffer from too many symmetries and supersymmetries, and appear content to stay in ten dimensions. Comparison with data requires an understanding, perhaps dynamical, of the breaking of these symmetries, for which we have no hint.

One ray of hope is afforded by the eleven-dimensional M-theory [1]. Although we know very little about it, save for its existence, it subsumes all known superstring theories, and may contain within it guideposts for supersymmetry-breaking and dimensional reduction. Its infrared limit is the much studied $N = 1$ supergravity in eleven dimensions [2], believed to be ultraviolet divergent.

Tracking and fixing divergences has proven to be a fruitful approach to physics, and fixing the divergences of $N = 1$ supergravity may well be a key to a deeper understanding of M-theory.

A technically difficult but conceptually simple framework for discussing divergences is the light-cone [3] frame formulation. In ten dimensions, the lack of divergences in string theories can be attributed to the triality of SO(8), the light-cone little group. The vector supermultiplet in ten dimensions with eight bosons and eight fermions produces, when dimensionally reduced, the $N = 4$ SuperYang-Mills theory in four dimensions, which is free of ultraviolet divergences [4, 5]. In 1982 Curtright [6] had in fact conjectured such a group theoretical relation between ultraviolet divergences and transverse little group representations for dimensionally-reduced theories. He applied a similar reasoning to eleven dimensions, and conjectured that the incomplete cancellation of the Dynkin indices of the SO(9) representations that describe $N = 1$ supergravity was responsible for the divergences of that theory.

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Interestingly, the relations among the Dynkin indices of the three representations of $N = 1$ Supergravity in eleven dimensions could be generalized to an infinite set of triplets of SO(9) representations, with the same incomplete cancellation among the eighth-order Dynkin indices [7]. Their mathematical origin was elucidated [8], and they were expressed in terms of light-cone superfields [9]. They describe massless fields of spin greater than two. Only with an infinite number of these Euler triplets can the well-known no-go theorems be evaded, pointing to a non-local generalization of $N = 1$ supergravity. To make this problem tractable, we first need to develop a light-cone description of $N = 1$ supergravity in eleven dimensions, and study its divergences. We intend to extend these calculations to include the Euler triplets.

This paper is a first step in this program. We study the simpler case of the SuperYang-Mills supermultiplet in ten dimensions, which is the massless sector of an open string theory. Starting from its dimensionally-reduced light-cone formulation, we show how to oxidize it to a fully ten-dimensional theory. Amazingly, the theory in four dimensions has so much memory of its original dimension that this can be done by simply introducing superspace generalizations of the transverse derivatives. The simplicity and compactness of the result makes us hopeful that it can be used to study all the symmetries of the ten-dimensional SuperYang-Mills theory.

Of course, the ten-dimensional theory is no longer ultraviolet finite, unless one adds the massive string states (a world-sheet approach to field theory such as in ref. [10] may be useful to develop this idea further). In future works, we aim to study how this taming of the field theory divergences comes about in the light-cone formalism. This ten-dimensional formulation may also be a stepping stone for developing the corresponding formalism for eleven-dimensional supergravity.

2. The LC_2 formulation of $N = 4$

Our starting point for this paper is the 1982 light-cone formulation of the $N = 4$ SuperYang-Mills theory [11], as originally described in ref [12]. This theory is the compactified version of SuperYang-Mills in ten dimensions, with one vector and one spinor field, both massless, linked by $N = 1$ supersymmetry, and members of the adjoint representation of a Lie algebra. In the unitary light-cone gauge with only physical degrees of freedom, they transform as the eight-dimensional representations (one bosonic, one fermionic) of SO(8), the transverse massless little group.

When this theory is dimensionally reduced to four dimensions, it consists of one complex bosonic field (the gauge field), four complex Grassmann fields and six scalars.

2.1 Field content

With the space-time metric $(-, +, +, \ldots, +)$, the light-cone coordinates and their derivatives are

$$x^{\pm} = \frac{1}{\sqrt{2}}\,(x^0 \pm x^3)\,; \qquad \partial^{\pm} = \frac{1}{\sqrt{2}}\,(-\partial_0 \pm \partial_3)\,; \tag{2.1}$$

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$$x = \frac{1}{\sqrt{2}}(x_1 + i\,x_2)\,; \qquad \bar\partial = \frac{1}{\sqrt{2}}(\partial_1 - i\,\partial_2)\,; \tag{2.2}$$

$$\bar x = \frac{1}{\sqrt{2}}(x_1 - i\,x_2)\,; \qquad \partial = \frac{1}{\sqrt{2}}(\partial_1 + i\,\partial_2)\,, \tag{2.3}$$

so that

$$\partial^+ x^- = \partial^- x^+ = -1\,; \qquad \bar\partial\, x = \partial\, \bar x = +1\,. \tag{2.4}$$

In four dimensions, any massless particle can be described by a complex field, and its complex conjugate of opposite helicity, the SO(2) coming from the little group decomposition

$$\mathrm{SO}(8) \supset \mathrm{SO}(2) \times \mathrm{SO}(6)\,. \tag{2.5}$$

Particles with no helicity are described by real fields. The eight vectors fields in ten dimension reduce to

$$\mathbf{8}_v = \mathbf{6}_0 + \mathbf{1}_1 + \mathbf{1}_{-1}\,, \tag{2.6}$$

and the eight spinors to

$$\mathbf{8}_s = \mathbf{4}_{1/2} + \bar{\mathbf{4}}_{-1/2}\,. \tag{2.7}$$

The representations on the right-hand side belong to $\mathrm{SO}(6) \sim \mathrm{SU}(4)$, with subscripts denoting the helicity: there are six scalar fields, two vector fields, four spinor fields and their conjugates. To describe them in a compact notation, we introduce anticommuting Grassmann variables θ^m and $\bar\theta_m$,

$$\{\theta^m\,,\,\theta^n\} = \{\bar\theta_m\,,\,\bar\theta_n\} = \{\bar\theta_m\,,\,\theta^n\} = 0\,, \tag{2.8}$$

which transform as the spinor representations of $\mathrm{SO}(6) \sim \mathrm{SU}(4)$,

$$\theta^m \sim \mathbf{4}_{1/2}\,; \qquad \overline{\theta}^m \sim \bar{\mathbf{4}}_{-1/2}\,, \tag{2.9}$$

where $m, n, p, q, \ldots = 1, 2, 3, 4$, denote SU(4) spinor indices. Their derivatives are written as

$$\bar\partial_m \equiv \frac{\partial}{\partial\,\theta^m}\,; \qquad \partial^m \equiv \frac{\partial}{\partial\,\bar\theta_m}\,, \tag{2.10}$$

with canonical anticommutation relations

$$\{\partial^m\,,\,\bar\theta_n\} = \delta^m{}_n\,; \qquad \{\bar\partial_m\,,\,\theta^n\} = \delta_m{}^n\,. \tag{2.11}$$

Under conjugation, upper and lower spinor indices are interchanged, so that $\overline{\theta^m} = \bar\theta_m$, while

$$\overline{(\bar\partial_m)} = -\partial^m\,; \qquad \overline{(\partial^m)} = -\,\bar\partial_m\,. \tag{2.12}$$

Also, the order of the operators is interchanged; that is $\overline{\theta^m\theta^n} = \bar\theta_n\,\bar\theta_m$, and $\overline{\partial^m\,\partial^n} = \bar\partial_n\,\bar\partial_m$.

All the physical degrees of freedom can be captured in one complex superfield

$$\begin{aligned}\phi(y) = {} & \frac{1}{\partial^+}\,A\,(y) + \frac{i}{\sqrt{2}}\,\theta^m\,\theta^n\,\overline{C}_{mn}\,(y) + \frac{1}{12}\,\theta^m\,\theta^n\,\theta^p\,\theta^q\,\epsilon_{mnpq}\,\partial^+\,\bar A\,(y) + \\ & + \frac{i}{\partial^+}\,\theta^m\,\bar\chi_m(y) + \frac{\sqrt{2}}{6}\theta^m\,\theta^n\,\theta^p\,\epsilon_{mnpq}\,\chi^q(y)\,. \end{aligned} \tag{2.13}$$

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In this notation, the eight original gauge fields $A_i\,, i = 1, \ldots, 8$ appear as

$$A = \frac{1}{\sqrt{2}}\,(A_1 + i\,A_2)\,, \qquad \bar{A} = \frac{1}{\sqrt{2}}\,(A_1 - i\,A_2)\,, \tag{2.14}$$

while the six scalar fields are written as antisymmetric SU(4) bi-spinors

$$C^{m\,4} = \frac{1}{\sqrt{2}}\,(A_{m+3} + i\,A_{m+6})\,, \qquad \overline{C}^{m\,4} = \frac{1}{\sqrt{2}}\,(A_{m+3} - i\,A_{m+6})\,, \tag{2.15}$$

for $m \neq 4$; complex conjugation is akin to duality,

$$\overline{C}_{mn} = \frac{1}{2}\,\epsilon_{mnpq}\,C^{pq}\,. \tag{2.16}$$

The fermion fields are denoted by χ^m and $\bar{\chi}_m$. All have adjoint indices (not shown here), and are local fields in the modified light-cone coordinates

$$y = \left(x,\, \bar{x},\, x^+,\, y^- \equiv x^- - \frac{i}{\sqrt{2}}\,\theta^m\,\bar{\theta}_m\right). \tag{2.17}$$

In this particular light-cone formulation, called LC_2 by some, all the unphysical degrees of freedom have been integrated out, leaving only the physical ones.

Introduce the chiral derivatives,

$$d^{\,m} = -\partial^m - \frac{i}{\sqrt{2}}\,\theta^m\,\partial^+\,; \qquad \bar{d}_{\,n} = \bar{\partial}_n + \frac{i}{\sqrt{2}}\,\bar{\theta}_n\,\partial^+\,, \tag{2.18}$$

which satisfy the anticommutation relations

$$\{d^m\,,\,\bar{d}_n\} = -i\,\sqrt{2}\,\delta^m{}_n\,\partial^+\,. \tag{2.19}$$

One verifies that ϕ and its complex conjugate $\bar{\phi}$ satisfy the chiral constraints

$$d^{\,m}\,\phi = 0\,; \qquad \bar{d}_{\,m}\,\bar{\phi} = 0\,, \tag{2.20}$$

as well as the "inside-out" constraints

$$\bar{d}_m\,\bar{d}_n\,\phi = \frac{1}{2}\,\epsilon_{mnpq}\,d^p\,d^q\,\bar{\phi}\,, \tag{2.21}$$

$$d^m\,d^n\,\bar{\phi} = \frac{1}{2}\,\epsilon^{mnpq}\,\bar{d}_p\,\bar{d}_q\,\phi\,. \tag{2.22}$$

The Yang-Mills action is then simply

$$\int d^4x \int d^4\theta\,d^4\bar{\theta}\,\mathcal{L}\,, \tag{2.23}$$

where

$$\begin{aligned}\mathcal{L} = &-\bar{\phi}\,\frac{\Box}{\partial^{+2}}\,\phi + \frac{4g}{3}\,f^{abc}\left(\frac{1}{\partial^+}\,\bar{\phi}^a\,\phi^b\,\bar{\partial}\,\phi^c + \text{complex conjugate}\right) - \\ &- g^2 f^{abc}\,f^{ade}\left(\frac{1}{\partial^+}(\phi^b\,\partial^+\phi^c)\frac{1}{\partial^+}\,(\bar{\phi}^d\,\partial^+\,\bar{\phi}^e) + \frac{1}{2}\,\phi^b\bar{\phi}^c\,\phi^d\,\bar{\phi}^e\right).\end{aligned} \tag{2.24}$$

Grassmann integration is normalized so that $\int d^4\theta\,\theta^1\theta^2\theta^3\theta^4 = 1$, and f^{abc} are the structure functions of the Lie algebra.

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2.2 Super-Poincaré algebra

The generators of the Poincaré algebra are given by the four momenta

$$p^- = -i\frac{\partial\bar\partial}{\partial^+}\,, \qquad p^+ = -i\,\partial^+\,, \qquad p = -i\,\partial\,, \qquad \bar p = -i\,\bar\partial\,, \tag{2.25}$$

using light-cone coordinates and no interactions; it also contains the kinematical transverse space rotations

$$j = x\,\bar\partial - \bar x\,\partial + S^{12}\,, \tag{2.26}$$

where the little group helicity generator is

$$S^{12} = \frac{1}{2}\,(\theta^p\,\bar\partial_p - \bar\theta_p\,\partial^p) + \frac{i}{4\sqrt{2}\,\partial^+}\,(d^p\,\bar d_p - \bar d_p\,d^p)\,. \tag{2.27}$$

This form is slightly different from the original expression of ref. [12] through the last term, which acts as a helicity counter. It also ensures that the chirality constraints are preserved

$$[j\,,\,d^m] = [j\,,\,\bar d_m] = 0\,. \tag{2.28}$$

Under an infinitesimal SO(2) transformation, this generator acts as a differential operator on the chiral superfield

$$\delta\,\phi = i\,\omega\,j\,\phi\,, \qquad \delta\,\bar\phi = -i\,\omega\,j\,\bar\phi\,. \tag{2.29}$$

The other kinematical generators are

$$j^+ = i\,x\,\partial^+\,, \qquad \bar j^+ = i\,\bar x\,\partial^+\,. \tag{2.30}$$

The rest of the generators must be specified separately for chiral and antichiral fields. Acting on ϕ, we have

$$j^{+-} = i\,x^-\,\partial^+ - \frac{i}{2}\,(\theta^p\bar\partial_p + \bar\theta_p\,\partial^p)\,, \tag{2.31}$$

chosen so as to preserve the chiral combination

$$[j^{+-}\,,\,y^-] = -i\,y^-\,, \tag{2.32}$$

and its commutators with the chiral derivatives

$$[j^{+-}\,,\,d^m] = \frac{i}{2}\,d^m\,, \qquad [j^{+-}\,,\,\bar d_m] = \frac{i}{2}\,\bar d_m\,, \tag{2.33}$$

preserve chirality. Similarly the dynamical boosts are

$$\begin{aligned} j^- &= i\,x\,\frac{\partial\bar\partial}{\partial^+} - i\,x^-\partial + i\left(\theta^p\bar\partial_p + \frac{i}{4\sqrt{2}\,\partial^+}\,(d^p\,\bar d_p - \bar d_p\,d^p)\right)\frac{\partial}{\partial^+}\,,\\ \bar j^- &= i\,\bar x\,\frac{\partial\bar\partial}{\partial^+} - i\,x^-\bar\partial + i\left(\bar\theta_p\partial^p + \frac{i}{4\sqrt{2}\,\partial^+}(d^p\,\bar d_p - \bar d_p\,d^p)\right)\frac{\bar\partial}{\partial^+}\,. \end{aligned} \tag{2.34}$$

They do not commute with the chiral derivatives,

$$[j^-\,,\,d^m] = \frac{i}{2}\,d^m\,\frac{\partial}{\partial^+}\,, \qquad [j^-\,,\,\bar d_m] = \frac{i}{2}\,\bar d_m\,\frac{\partial}{\partial^+}\,, \tag{2.35}$$

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but do not change the chirality of the fields on which they act. They satisfy the Poincaré algebra, in particular

$$[j^-\,,\,\bar{j}^+] = -i\,j^{+-} - j\,, \qquad [j^-\,,\,j^{+-}] = i\,j^-\,. \tag{2.36}$$

On the light-cone, supersymmetry breaks up into two types, kinematical and dynamical. The kinematical supersymmetries

$$q_+^m = -\partial^m + \frac{i}{\sqrt{2}}\,\theta^m\,\partial^+\,; \qquad \bar{q}_{+n} = \bar{\partial}_n - \frac{i}{\sqrt{2}}\,\bar{\theta}_n\,\partial^+\,, \tag{2.37}$$

satisfy

$$\{q_+^m\,,\,\bar{q}_{+n}\} = i\,\sqrt{2}\,\delta^m{}_n\,\partial^+\,. \tag{2.38}$$

and anticommute with the chiral derivatives

$$\{q_+^m\,,\,\bar{d}_n\} = \{d^m\,,\,\bar{q}_{+n}\} = 0\,. \tag{2.39}$$

The dynamical supersymmetries are obtained by boosting the kinematical ones

$$q_-^m \equiv i\,[\bar{j}^-\,,\,q_+^m] = \frac{\partial}{\partial^+}\,q_+^m\,, \qquad \bar{q}_{-m} \equiv i\,[j^-\,,\,\bar{q}_{+m}] = \frac{\bar{\partial}}{\partial^+}\,\bar{q}_{+m}\,. \tag{2.40}$$

They satisfy the free $N = 4$ supersymmetry algebra

$$\{q_-^m\,,\,\bar{q}_{-n}\} = i\,\sqrt{2}\,\delta^m{}_n\,\frac{\partial\bar{\partial}}{\partial^+}\,. \tag{2.41}$$

In the interacting theory, all dynamical generators will be altered by interactions, as discussed in ref. [13].

3. Ten dimensions

The very compact formalism of the previous section was constructed for the $N = 4$ theory in four dimensions. In this paper we generalize this formalism to restore the theory in ten dimensions, without changing the superfield, simply by introducing generalized derivative operators.

First of all, the transverse light-cone variables need to be generalized to eight. We stick to the previous notation, and introduce the six extra coordinates and their derivatives as antisymmetric bi-spinors

$$x^{m\,4} = \frac{1}{\sqrt{2}}\,(x_{m+3} + i\,x_{m+6})\,, \qquad \partial^{m\,4} = \frac{1}{\sqrt{2}}\,(\partial_{m+3} + i\,\partial_{m+6})\,, \tag{3.1}$$

for $m \neq 4$, and their complex conjugates

$$\bar{x}_{pq} = \frac{1}{2}\,\epsilon_{pqmn}\,x^{mn}\,; \qquad \bar{\partial}_{pq} = \frac{1}{2}\,\epsilon_{pqmn}\,\partial^{mn}\,. \tag{3.2}$$

Their derivatives satisfy

$$\bar{\partial}_{mn}\,x^{pq} = (\delta_m{}^p\,\delta_n{}^q - \delta_m{}^q\,\delta_n{}^p)\,; \qquad \partial^{mn}\,\bar{x}_{pq} = (\delta^m{}_p\,\delta^n{}_q - \delta^m{}_q\,\delta^n{}_p)\,, \tag{3.3}$$

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and

$$\partial^{mn}\, x^{pq} = \frac{1}{2}\,\epsilon^{pqrs}\,\partial^{mn}\,\bar{x}_{rs} = \epsilon^{mnpq}\,. \tag{3.4}$$

There are no modifications to be made to the chiral superfield, except for the dependence on the extra coordinates

$$A(y) = A(x, \bar{x}, x^{mn}, \bar{x}_{mn}, y^{-})\,, \qquad \text{etc}\,\ldots \tag{3.5}$$

These extra variables will be acted on by new operators that generate the higher-dimensional symmetries.

3.1 The super-Poincaré algebra in 10 dimensions

The SuperPoincaré algebra needs to be generalized from the form in ref. [12]. One starts with the construction of the SO(8) little group using the decomposition SO(8) $\supset$ SO(2) $\times$ SO(6). The SO(2) generator is the same; the SO(6) $\sim$ SU(4) generators are given by

$$\begin{aligned} J^{m}{}_{n} = {} & \frac{1}{2}\,(x^{mp}\,\bar{\partial}_{pn} - \bar{x}_{pn}\,\partial^{mp}) - \theta^{m}\,\bar{\partial}_{n} + \bar{\theta}_{n}\,\partial^{m} + \frac{1}{4}\,(\theta^{p}\,\bar{\partial}_{p} - \bar{\theta}_{p}\,\partial^{p})\,\delta^{m}{}_{n} + \\ & + \frac{i}{2\sqrt{2}\,\partial^{+}}\,(d^{m}\,\bar{d}_{n} - \bar{d}_{n}\,d^{m}) + \frac{i}{8\sqrt{2}\,\partial^{+}}\,(d^{p}\,\bar{d}_{p} - \bar{d}_{p}\,d^{p})\,\delta^{m}{}_{n}\,. \end{aligned} \tag{3.6}$$

The extra terms with the d and $\bar{d}$ operators are not necessary for closure of the algebra. However they insure that the generators commute with the chiral derivatives. They satisfy the commutation relations

$$[J\,,\, J^{m}{}_{n}] = 0\,, \qquad [J^{m}{}_{n}\,,\, J^{p}{}_{q}] = \delta^{m}{}_{q}\,J^{p}{}_{n} - \delta^{p}{}_{n}\,J^{m}{}_{q}\,. \tag{3.7}$$

The remaining SO(8) generators lie in the coset SO(8)/(SO(2) $\times$ SO(6))

$$\begin{aligned} J^{pq} &= x\,\partial^{pq} - x^{pq}\,\partial + \frac{i}{\sqrt{2}}\,\partial^{+}\,\theta^{p}\,\theta^{q} - i\,\sqrt{2}\,\frac{1}{\partial^{+}}\,\partial^{p}\,\partial^{q} + \frac{i}{\sqrt{2}\,\partial^{+}}\,d^{p}\,d^{q}\,, \\ \bar{J}_{mn} &= \bar{x}\,\bar{\partial}_{mn} - \bar{x}_{mn}\,\bar{\partial} + \frac{i}{\sqrt{2}}\,\partial^{+}\,\bar{\theta}_{m}\,\bar{\theta}_{n} - i\,\sqrt{2}\,\frac{1}{\partial^{+}}\,\bar{\partial}_{m}\,\bar{\partial}_{n} + \frac{i}{\sqrt{2}\,\partial^{+}}\,\bar{d}_{m}\,\bar{d}_{n}\,. \end{aligned} \tag{3.8}$$

All SO(8) transformations are specially constructed so as not to mix chiral and antichiral superfields,

$$[J^{mn}\,,\, \bar{d}_{p}] = 0\,; \qquad [\bar{J}_{mn}\,,\, d^{p}] = 0\,, \tag{3.9}$$

and satisfy the SO(8) commutation relations

$$\begin{aligned} [J\,,\, J^{mn}] &= J^{mn}\,, \qquad [J\,,\, \bar{J}_{mn}] = -\bar{J}_{mn}\,, \\ [J^{m}{}_{n}\,,\, J^{pq}] &= \delta^{q}{}_{n}\,J^{mp} - \delta^{p}{}_{n}\,J^{mq}\,, \qquad [J^{m}{}_{n}\,,\, \bar{J}_{pq}] = \delta^{m}{}_{q}\,\bar{J}_{np} - \delta^{m}{}_{p}\,\bar{J}_{nq}\,, \\ [J^{mn}\,,\, \bar{J}_{pq}] &= \delta^{m}{}_{p}J^{n}{}_{q} + \delta^{n}{}_{q}J^{m}{}_{p} - \delta^{n}{}_{p}J^{m}{}_{q} - \delta^{m}{}_{q}J^{n}{}_{p} - (\delta^{m}{}_{p}\,\delta^{n}{}_{q} - \delta^{n}{}_{p}\,\delta^{m}{}_{q})\,J\,. \end{aligned}$$

Rotations between the 1 or 2 and 4 through 9 directions induce on the chiral fields the changes

$$\delta\,\phi = \left(\frac{1}{2}\,\omega_{mn}\,J^{mn} + \frac{1}{2}\,\bar{\omega}^{mn}\,\bar{J}_{mn}\right)\phi\,, \tag{3.10}$$

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where complex conjugation is like duality

$$\bar{\omega}_{pq} = \frac{1}{2}\,\epsilon_{mnpq}\,\omega^{mn}\,. \tag{3.11}$$

For example, a rotation in the $1-4$ plane through an angle θ corresponds to taking $\theta = \omega_{14} = \omega_{23}$ $(= \omega^{23} = \omega^{14}$ by reality), all other components being zero. Finally, we verify that the kinematical supersymmetries are duly rotated by these generators

$$[J^{mn}\,,\,\bar{q}_{+\,p}] = \delta^{n}{}_{p}\,q_{+}^{m} - \delta^{m}{}_{p}\,q_{+}^{n}\,; \qquad [\bar{J}_{mn}\,,\,q_{+}^{p}] = \delta_{n}{}^{p}\,\bar{q}_{+\,m} - \delta_{m}{}^{p}\,\bar{q}_{+\,n}\,. \tag{3.12}$$

We now use the SO(8) generators to construct the super-Poincaré generators

$$\begin{aligned} J^{+} &= i\,x\,\partial^{+}\,; & \bar{J}^{+} &= i\,\bar{x}\,\partial^{+} \\ J^{+\,mn} &= i\,x^{mn}\,\partial^{+}\,; & \bar{J}^{+}{}_{mn} &= i\,\bar{x}_{mn}\,\partial^{+}\,. \end{aligned} \tag{3.13}$$

The dynamical boosts are now

$$\begin{aligned} J^{-} = {} & i\,x\,\frac{\partial\bar{\partial} + \frac{1}{4}\,\bar{\partial}_{pq}\,\partial^{pq}}{\partial^{+}} - i\,x^{-}\,\partial + i\,\frac{\partial}{\partial^{+}}\Big\{\theta^{m}\,\bar{\partial}_{m} + \frac{i}{4\sqrt{2}\,\partial^{+}}\,(d^{p}\,\bar{d}_{p} - \bar{d}_{p}\,d^{p})\Big\} - \\ & - \frac{1}{4}\,\frac{\bar{\partial}_{pq}}{\partial^{+}}\Big\{\frac{\partial^{+}}{\sqrt{2}}\,\theta^{p}\,\theta^{q} - \frac{\sqrt{2}}{\partial^{+}}\,\partial^{p}\,\partial^{q} + \frac{1}{\sqrt{2}\partial^{+}}d^{p}\,d^{q}\Big\}, \end{aligned} \tag{3.14}$$

and its conjugate

$$\begin{aligned} \bar{J}^{-} = {} & i\,\bar{x}\,\frac{\partial\bar{\partial} + \frac{1}{4}\,\bar{\partial}_{pq}\,\partial^{pq}}{\partial^{+}} - i\,x^{-}\,\bar{\partial} + i\,\frac{\bar{\partial}}{\partial^{+}}\Big\{\bar{\theta}_{m}\,\partial^{m} + \frac{i}{4\sqrt{2}\,\partial^{+}}\,(d^{p}\,\bar{d}_{p} - \bar{d}_{p}\,d^{p})\Big\} - \\ & - \frac{1}{4}\,\frac{\partial^{pq}}{\partial^{+}}\Big\{\frac{\partial^{+}}{\sqrt{2}}\,\bar{\theta}_{p}\,\bar{\theta}_{q} - \frac{\sqrt{2}}{\partial^{+}}\,\bar{\partial}_{p}\,\bar{\partial}_{q} + \frac{1}{\sqrt{2}\partial^{+}}\bar{d}_{p}\,\bar{d}_{q}\Big\}. \end{aligned} \tag{3.15}$$

The others are obtained by using the $\mathrm{SO}(8)/(\mathrm{SO}(2)\times\mathrm{SO}(6))$ rotations

$$J^{-\,mn} = [J^{-}\,,\,J^{mn}]\,; \qquad \bar{J}^{-}{}_{mn} = [\bar{J}^{-}\,,\,\bar{J}_{mn}]\,. \tag{3.16}$$

We do not show their explicit forms as they are too cumbersome. The four supersymmetries in four dimensions turn into one supersymmetry in ten dimensions. In our notation, the kinematical supersymmetries q_{+}^{n} and $\bar{q}_{+n}$, are assembled into one SO(8) spinor. The dynamical supersymmetries are obtained by boosting

$$i\,[\bar{J}^{-}\,,\,q_{+}^{m}] \equiv \mathcal{Q}^{m}\,, \qquad i\,[J^{-}\,,\,\bar{q}_{+\,m}] \equiv \overline{\mathcal{Q}}_{m}\,, \tag{3.17}$$

where

$$\begin{aligned} \mathcal{Q}^{m} &= \frac{\bar{\partial}}{\partial^{+}}\,q_{+}{}^{m} + \frac{1}{2}\,\frac{\partial^{mn}}{\partial^{+}}\,\bar{q}_{+\,n}\,, \\ \overline{\mathcal{Q}}_{m} &= \frac{\partial}{\partial^{+}}\,\overline{q}_{+\,m} + \frac{1}{2}\,\frac{\overline{\partial}_{mn}}{\partial^{+}}\,q_{+}^{\,n}\,. \end{aligned} \tag{3.18}$$

They satisfy the supersymmmetry algebra

$$\{\mathcal{Q}^{\,m}\,,\,\overline{\mathcal{Q}}_{\,n}\} = i\,\sqrt{2}\,\delta^{m}_{\ n}\,\frac{1}{\partial^{+}}\Big(\partial\,\overline{\partial} + \frac{1}{4}\,\overline{\partial}_{pq}\,\partial^{pq}\Big)\,, \tag{3.19}$$

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and can be obtained from one another by SO(8) rotations, as

$$\frac{1}{2}\,\epsilon_{pqmn}\,[J^{pq}\,,\;\mathcal{Q}^{\,m}] = 4\,\overline{\mathcal{Q}}_n\,, \tag{3.20}$$

while

$$[\bar{J}_{pq}\,,\;\mathcal{Q}^{\,m}] = 0\,. \tag{3.21}$$

Note also that

$$\{\mathcal{Q}^{\,m}\,,\;q_+^n\} = \frac{i}{\sqrt{2}}\,\partial^{mn}\,. \tag{3.22}$$

We thank the anonymous referee for suggesting that we look at central charges in this framework. It provides a simple method to introduce the central charges germane to the four-dimensional theory: the six-dimensional derivatives ∂^{mn}, are simply replaced by c-numbers Z^{mn}, thus yielding the massive supersymmetry algebra in four dimensions with six central charges,

$$Z^{mn} = \frac{1}{2}\,\epsilon^{mnpq}\,\overline{Z}_{pq}\,.$$

However, relaxing this duality condition, gives us all 12 central charges of the $N = 4$ Yang-Mills Theory, thus indicating that both varieties of central charges have a common origin in the light-cone gauge formulation.

Starting from these results for the $N = 1$ SuperPoincaré generators for the free theory, we proceed to build the interacting theory in ten dimensions.

3.2 The generalized derivatives

The cubic interaction in the $N = 4$ lagrangian contains explicitly the derivative operators ∂ and $\bar{\partial}$. To achieve covariance in ten dimensions, these must be generalized. We propose the following operator

$$\overline{\nabla} \equiv \bar{\partial} + \frac{i\,\alpha}{4\,\sqrt{2}\,\partial^+}\,\bar{d}_p\,\bar{d}_q\,\partial^{pq}\,, \tag{3.23}$$

which naturally incorporates the rest of the derivatives ∂^{pq}, with α as an arbitrary parameter. After some algebra, we find that $\overline{\nabla}$ is covariant under SO(8) transformations. We define its rotated partner as

$$\nabla^{mn} \equiv \left[\overline{\nabla}\,,\;J^{mn}\right], \tag{3.24}$$

where

$$\nabla^{mn} = \partial^{mn} - \frac{i\,\alpha}{4\sqrt{2}\,\partial^+}\bar{d}_r\,\bar{d}_s\,\epsilon^{mnrs}\,\partial\,. \tag{3.25}$$

If we apply to it the inverse transformation, it goes back to the original form

$$\left[\overline{J}_{pq}\,,\;\nabla^{mn}\right] = (\delta_p{}^m\,\delta_q{}^n\; - \delta_q{}^m\,\delta_p{}^n)\,\overline{\nabla}\,, \tag{3.26}$$

and these operators transform under SO(8)/(SO(2) × SO(6)), and SO(2) × SO(6) as the components of an 8-vector.

We introduce the conjugate operator ∇ by requiring that

$$\nabla\,\bar{\phi} \equiv \overline{(\overline{\nabla}\,\phi)}\,, \tag{3.27}$$

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with

$$\nabla \equiv \partial + \frac{i\,\alpha}{4\sqrt{2}\,\partial^+}\, d^p\, d^q\, \bar\partial_{pq}\,. \tag{3.28}$$

Define

$$\overline{\nabla}_{mn} \equiv \left[\nabla\,,\, \bar J_{mn}\right], \tag{3.29}$$

which is given by

$$\overline{\nabla}_{mn} = \bar\partial_{mn} - \frac{i\,\alpha}{4\sqrt{2}\,\partial^+}\, d^r\, d^s\, \epsilon_{mnrs}\,\bar\partial\,. \tag{3.30}$$

We then verify that

$$\left[J^{mn}\,,\, \overline{\nabla}_{pq}\right] = (\delta_p{}^m\,\delta_q{}^n\, - \delta_q{}^m\,\delta_p{}^n)\,\nabla\,. \tag{3.31}$$

The value of the parameter α is fixed by the invariance of the cubic interaction.

3.3 Invariance of the action

The kinetic term is trivially made SO(8)-invariant by including the six extra transverse derivatives in the d'Alembertian. The quartic interactions are obviously invariant since they do not contain any transverse derivative operators. Hence we need only consider the cubic vertex.

The point of our paper is that to achieve covariance in ten dimensions, it suffices to replace the transverse ∂ and $\bar\partial$ by ∇ and $\overline{\nabla}$, respectively. We propose the new cubic interaction term

$$\frac{4g}{3}\, f^{abc} \int d^{10}x \int d^4\,\theta\, d^4\,\bar\theta \left(\frac{1}{\partial^+}\,\bar\phi^a\,\phi^b\,\overline{\nabla}\,\phi^c + \text{complex conjugate}\right) \tag{3.32}$$

Since it is obviously invariant under SO(6) × SO(2), we need only consider the coset variations. On the chiral superfield, using the chiral constraints,

$$\begin{aligned}
\delta_J\,\phi &\equiv \overline{\omega}_{mn}\, J^{mn}\,\phi = i\,\sqrt{2}\,\overline{\omega}_{mn}\,\partial^+\,\theta^m\,\theta^n\,\phi \\
\delta_{\bar J}\,\phi &\equiv \omega^{pq}\,\overline{J}_{pq}\,\phi = \omega^{pq}\left\{\frac{i}{\sqrt{2}}\,\partial^+\,\bar\theta_p\,\bar\theta_q - i\,\sqrt{2}\,\frac{1}{\partial^+}\,\bar\partial_p\,\bar\partial_q + \frac{i}{\sqrt{2}}\,\frac{1}{\partial^+}\,\overline{d}_p\,\overline{d}_q\right\}\phi\,.
\end{aligned} \tag{3.33}$$

We list the conjugate relations for completeness

$$\begin{aligned}
\delta_{\bar J}\,\overline{\phi} &= \omega^{pq}\,\overline{J}_{pq}\,\overline{\phi} = i\,\sqrt{2}\;\;\omega^{pq}\,\partial^+\,\bar\theta_p\,\bar\theta_q\,\overline{\phi}\,, \\
\delta_J\,\overline{\phi} &= \overline{\omega}_{mn}\, J^{mn}\,\overline{\phi} = \overline{\omega}_{mn}\left\{\frac{i}{\sqrt{2}}\,\partial^+\,\theta^m\,\theta^n\, - i\,\sqrt{2}\,\frac{1}{\partial^+}\,\partial^m\,\partial^n + \frac{i}{\sqrt{2}}\,\frac{1}{\partial^+}\,d^m\,d^n\right\}\overline{\phi}\,.
\end{aligned} \tag{3.34}$$

The variations of the generalized derivative and its conjugate are given by

$$\begin{aligned}
\delta_J\,\overline{\nabla} &= \overline{\omega}_{mn}\,[J^{mn}\,,\,\overline{\nabla}] = -\overline{\omega}_{mn}\,\nabla^{mn}\,, \\
\delta_{\bar J}\,\overline{\nabla} &= \omega^{pq}\,[\overline{J}_{pq}\,,\,\overline{\nabla}] = \frac{i\,\alpha}{2\,\sqrt{2}}\,\omega^{pq}\bar d_p\bar d_q\,\frac{\overline{\partial}}{\partial^+}\,.
\end{aligned} \tag{3.35}$$

Invariance under SO(8) is checked by doing a δ_J variation on the cubic vertex, *including* its complex conjugate. In terms of

$$\mathcal{V} + \overline{\mathcal{V}} \equiv f_{abc}\int\left(\frac{1}{\partial^+}\overline{\phi}^a\,\phi^b\overline{\nabla}\;\phi^c + \frac{1}{\partial^+}\phi^a\overline{\phi}^b\nabla\overline{\phi}^c\right), \tag{3.36}$$

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explicit calculations yield

$$\delta_J \mathcal{V} = f_{abc}\,\overline{\omega}_{mn} \int \left(\left(\frac{\alpha}{2} - 1\right) \frac{1}{\partial^+}\,\overline{\phi}^a\,\phi^b \partial^{mn}\,\phi^c + \frac{i\,\alpha}{2\,\sqrt{2}}\,\frac{1}{\partial^+}\,\overline{\phi}^a\,\phi^b d^m\,d^n\,\frac{\partial}{\partial^+}\,\overline{\phi}^c \right), \tag{3.37}$$

and

$$\begin{aligned} \delta_J\,\overline{\mathcal{V}} &= \overline{\delta_{\bar{J}}\,\mathcal{V}} \\ &= f_{abc}\,\overline{\omega}_{mn} \int \left(-\frac{\alpha}{2}\,\phi^a\,\frac{1}{\partial^+}\,\overline{\phi}^b \partial^{mn}\,\phi^c + \frac{i}{\sqrt{2}}\,\phi^a\,\frac{1}{\partial^+}\,\overline{\phi}^b d^m\,d^n\,\frac{\partial}{\partial^+}\,\overline{\phi}^c \right. \\ &\qquad \left. + \frac{i\,\alpha}{2\,\sqrt{2}}\,\frac{1}{\partial^+}\,\phi^a \overline{\phi}^b\,d^m\,d^n\,\frac{\partial}{\partial^+}\,\overline{\phi}^c \right). \end{aligned} \tag{3.38}$$

Adding the two yields the final result

$$\delta_J\,(\mathcal{V} + \overline{\mathcal{V}}) = (\alpha - 1)\,f_{abc}\,\overline{\omega}_{mn} \int \left(\frac{1}{\partial^+}\,\overline{\phi}^a\,\phi^b \partial^{mn}\,\phi^c + \frac{i}{\sqrt{2}\,\partial^+}\,\overline{\phi}^a \phi^b d^m\,d^n\,\frac{\partial}{\partial^+}\,\overline{\phi}^c \right). \tag{3.39}$$

The cubic vertex is SO(8) invariant if $\alpha = 1$, and the generalized derivative is totally determined

$$\overline{\nabla} = \bar{\partial} + \frac{i}{4\,\sqrt{2}\,\partial^+}\,\bar{d}_p\,\bar{d}_q\,\partial^{pq}\,. \tag{3.40}$$

To be sure, we have checked invariance by performing the Grassmann integrations, and looking at the components.

To obtain this result, we have used the antisymmetry of the structure functions, the chiral constraints, the "inside-out" constraints, and performed integrations by parts on the coordinates and Grassmann variables. In particular, using the relation between the chiral field and its conjugate, implied by the "inside out" constraints

$$\overline{\phi} = \frac{1}{2 \cdot 4!}\,\epsilon^{pqmn}\,\bar{d}_p\,\bar{d}_q\,\bar{d}_m\,\bar{d}_n\,\frac{1}{\partial^{+2}}\,\phi\,, \tag{3.41}$$

we deduce two magical identities

$$f_{abc} \int \frac{1}{\partial^{+2}}\,\overline{\phi}^a\,\phi^b \bar{\partial}\,\phi^c = 0\,, \tag{3.42}$$

$$f_{abc} \int \frac{1}{\partial^+}\,\overline{\phi}^a\,\frac{1}{\partial^+}\,\phi^b d^m\,d^n\,\partial\,\overline{\phi}^c = 0\,. \tag{3.43}$$

In this light-cone form, the Lorentz invariance in ten dimensions is automatic once the little group invariance has been established. We have therefore shown ten-dimensional invariance, since the quartic term does not need to be changed.

4. Conclusions

We have explicitly constructed the light-cone action of ten-dimensional SuperYang-Mills, starting from its four-dimensional realization. Crucial to the construction is the generalized derivative whose components form an SO(8) vector. Although its geometrical meaning is not entirely clear to us, it suggests that this action has further hidden symmetries, an issue we hope to come back to in the future.

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As we stated in the introduction, this oxidation of the light-cone formulation of the $N = 4$ theory should naturally lead us to a similar phenomenon for $N = 8$ supergravity in four dimensions. In that case, we have to add in the seven missing coordinates and the new derivatives should look very much like the one in (63), with SO(7) substituted for SO(6). In a future publication [14], we will show how it leads along similar lines to $N = 1$ Supergravity in eleven dimensions.

References

[1] C.M. Hull and P.K. Townsend, *Unity of superstring dualities*, *Nucl. Phys.* **B 438** (1995) 109 [hep-th/9410167];
E. Witten, *String theory dynamics in various dimensions*, *Nucl. Phys.* **B 443** (1995) 85 [hep-th/9503124].

[2] E. Cremmer, B. Julia and J. Scherk, *Supergravity theory in 11 dimensions*, *Phys. Lett.* **B 76** (1978) 409.

[3] P.A.M. Dirac, *Forms of relativistic dynamics*, *Rev. Mod. Phys.* **21** (1949) 392.

[4] L. Brink, O. Lindgren and B.E.W. Nilsson, *The ultraviolet finiteness of the $N = 4$ Yang-Mills theory*, *Phys. Lett.* **B 123** (1983) 323.

[5] S. Mandelstam, *Light cone superspace and the ultraviolet finiteness of the $N = 4$ model*, *Nucl. Phys.* **B 213** (1983) 149.

[6] T.L. Curtright, *Indices, triality and ultraviolet divergences for supersymmetric theories*, *Phys. Rev. Lett.* **48** (1982) 1704.

[7] T. Pengpan and P. Ramond, *M(ysterious) patterns in* SO(9), *Phys. Rept.* **315** (1999) 137 [hep-th/9808190].

[8] B. Gross, B. Kostant, P. Ramond and S. Sternberg, *THe Weyl character formula, the half spin representations, and equal rank subgroups*, *Proc. Nat. Acad. Sci.* **95** (1998) 8441 [math.RT/9808133].

[9] L. Brink, P. Ramond and X.-Z. Xiong, *Supersymmetry and Euler multiplets*, *J. High Energy Phys.* **10** (2002) 058 [hep-th/0207253].

[10] K. Bardakci and C.B. Thorn, *A worldsheet description of large-N_c quantum field theory*, *Nucl. Phys.* **B 626** (2002) 287 [hep-th/0110301].

[11] L. Brink, J.H. Schwarz and J. Scherk, *Supersymmetric Yang-Mills theories*, *Nucl. Phys.* **B 121** (1977) 77;
F. Gliozzi, J. Scherk and D.I. Olive, *Supersymmetry, supergravity theories and the dual spinor model*, *Nucl. Phys.* **B 122** (1977) 253.

[12] L. Brink, O. Lindgren and B.E.W. Nilsson, *$N = 4$ Yang-Mills theory on the light cone*, *Nucl. Phys.* **B 212** (1983) 401.

[13] A.K.H. Bengtsson, I. Bengtsson and L. Brink, *Cubic interaction terms for arbitrarily extended supermultiplets*, *Nucl. Phys.* **B 227** (1983) 41.

[14] S. Ananth, L. Brink and P. Ramond, in preparation.

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